

Asymptotic estimates of the distribution of Brownian hitting time of a disc

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Abstract

The distribution of the first hitting time of a disc for the standard two dimensional Brownian motion is computed. By investigating the inversion integral of its Laplace transform we give fairly detailed asymptotic estimates of its density valid uniformly with respect to the point where the Brownian motion starts from.

1 Introduction

Let B_t be the standard two-dimensional Brownian motion and $\mathbf{t}_r^{(2)}$ the Brownian first hitting time of the disc of radius $r > 0$ centered at the origin: $\mathbf{t}_r^{(2)} = \inf\{t > 0 : |B_t| \leq r\}$. In this paper we obtain precise asymptotic estimates of the distribution function $P_x[\mathbf{t}_r^{(2)} > t]$ and its density

$$p_{r,x}(t) = P_x[\mathbf{t}_r^{(2)} \in dt]/dt,$$

where P_x denotes the probability law of $(B_t)_{t \geq 0}$ started at x . $p_{r,x}(t)$ is invariant under rotation of x and the function $u(t, \xi) = p_{r,(\xi,0)}(t)$ solves the diffusion equation $u_t = \frac{1}{2}u_{\xi\xi} + (2\xi)^{-1}u_{\xi}$ together with the initial-boundary conditions $u(0, \xi) = 0$ ($\xi > r$) and $u(\cdot, r) = \delta$ (Dirac's delta function). The corresponding interior problem of estimating the exit time distribution from a circle, which admits a quite nice eigenfunction expansion, is well understood and completely different from the present problem.

An explicit form of the Laplace transform of $p_{r,x}$ is well known (see (7) below) and its inversion gives

$$p_{r,x}(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{K_0(|x|\sqrt{2z})}{K_0(r\sqrt{2z})} e^{tz} dz, \quad |x| > r. \quad (1)$$

Although much information for the distribution of $\mathbf{t}_r^{(2)}$ can be obtained directly from the property of the Laplace transform itself (an interesting illustration of this is found in [3], problem 4.6.4, where the limit law of $\lg \mathbf{t}_r^{(2)} / \lg r^{-1}$ as $r \rightarrow 0$ is found in a very simple way), here we wish to find more or less explicit asymptotic expressions of $p_{r,x}(t)$ as $t \rightarrow \infty$ (such an expression for $\int_0^t p_{r,x}(s) ds$ is derived from (1) in [5], a weak version of our Theorem 3, in the case when x^2/t is constant). The motivation for the latter comes from the corresponding problem for the random walks for which the results of this paper are useful (see Remark 3 below).

We derive two asymptotic estimates of $p_{r,x}(t)$, one sharp when $|x|^2/t$ is small and the other so when both x and t are large. For the former case the result is given in terms of the function

$$W(\lambda) = \int_0^\infty \frac{e^{-\lambda u} du}{[\lg u]^2 + \pi^2} \quad (\lambda > 0). \quad (2)$$

We fix a positive constant r_o once for all and put $c_o = -2\gamma + \lg(2/r_o^2)$ so that $\frac{1}{2}r_o^2 = e^{-(c_o+2\gamma)}$, where $\gamma = -\int_0^\infty (\lg u)e^{-u}du$ (Euler's constant).

We write $a \vee b = \max\{a, b\}$, $a \wedge b = \min\{a, b\}$, $\lg^+ a = \lg(a \vee 1)$ for a, b real and $x^2 = |x|^2$.

Theorem 1 *Uniformly for $|x| > r_o$, as $t \rightarrow \infty$*

$$p_{r_o, x}(t) = 2 \left[\lg \frac{|x|}{r_o} \right] e^{c_o} W(e^{c_o} t) + O\left(\frac{1 + \lg^+ |x|}{t(\lg t)^2} \cdot \frac{x^2 \wedge t}{t} \right). \quad (3)$$

REMARK 1. The function $\lambda W(\lambda)$ admits the following asymptotic expansion in powers of $1/\lg \lambda$:

$$\lambda W(\lambda) = \frac{1}{(\lg \lambda)^2} - \frac{2\gamma}{(\lg \lambda)^3} - \frac{\frac{1}{2}\pi^2 - 3\gamma^2}{(\lg \lambda)^4} + \dots \quad (4)$$

valid in the both limits as $\lambda \rightarrow \infty$ and as $\lambda \downarrow 0$ (cf. [1]). In Appendix we shall indicate how to derive this expansion; therein one will also find three Fourier representations of W . In [6] the present author uses the function $N(\lambda) := \int_\lambda^\infty W(t)dt$ to express the leading term of the asymptotic formula of the mean number of visited sites of random walk on \mathbf{Z}^2 ; $N(\lambda)$ is called the Ramanujan function in [8] and [1], where asymptotic expansions of a class of functions including it are obtained.

For $|x|$ much smaller than \sqrt{t} , Theorem 1 provides an effective asymptotic expansion of $p_{r_o}(t, x)$ in powers of $1/\lg(e^{c_o}t)$ in view of (4), while in the region where $|x|$ is comparable with \sqrt{t} the estimate of Theorem 1 is poor. In the latter case, i.e. the case when x^2/t is bounded away from both zero and infinity, the next theorem provides such an estimate that the ratio of the error term to the leading term is not $O(1/(\lg t))$ but at most $O(1/(\lg t)^2)$, a smaller order of magnitude than one that may be expected from Theorem 1.

Theorem 2 *Uniformly for $|x| > r_o$, as $t \rightarrow \infty$*

$$p_{r_o, x}(t) = \frac{\lg(\frac{1}{2}e^{c_o}x^2)}{t(\lg(e^{c_o}t))^2} e^{-x^2/2t} + \begin{cases} \frac{2\gamma \lg(t/x^2)}{t(\lg t)^3} + O\left(\frac{1}{t(\lg t)^3}\right) & \text{for } x^2 < t \\ O\left(\frac{1 + [\lg(x^2/t)]^2}{x^2(\lg t)^3}\right) & \text{for } x^2 \geq t. \end{cases} \quad (5)$$

REMARK 2. Substituting from the identity $1/r_o = (\frac{1}{2}e^{c_o})^{1/2}e^\gamma$ into the first term on the right side of (3) one may realize that Theorem 1 implies (5) if restricted to the region $x^2 \leq Ct/(\lg t)^2$. The error estimate in (5) is best possible in any region where x^2/t is bounded, while it can be improved significantly for x^2/t large (enough if larger roughly than $4 \lg \lg t$) (see Remark 5 given at the end of Section 3).

REMARK 3. For a random walk on the two-dimensional square lattice \mathbf{Z}^2 we have analogues of Theorems 1 and 2 [7]. Suppose that the walk is aperiodic and satisfies $EX = 0$, $E|X^{2+\delta}| < \infty$ for some $\delta \in [0, 2]$. Let $a(x)$ denote the potential function of the walk and $f_x(k)$ the probability that the random walk started from x hits the origin for the first time at the k -th step. Then, uniformly in $x \in \mathbf{Z}^2$, as $k \rightarrow \infty$

$$f_x(k) = 2\pi |Q|^{1/2} a^*(x) e^{c_o} W(e^{c_o} k) \left[1 + o(k^{-\delta/2}) \right] + O\left(\frac{|x|_+^2}{k^2 \lg k} \right),$$

where $a^*(x) = \delta_{0,x} + a(x)$ and c_o is a certain constant depending on the walk. As to Theorem 2 the very analogue of (5) holds for $f_x(k)$ (one may simply replace t by k on its right side), provided $\delta > 0$.

One can readily obtain an estimate of $P_x[\mathbf{t}_{r_o}^{(2)} < t] = \int_0^t p_{r_o, x}(u) du$ from Theorem 1 that is relatively sharp if $x^2 < t/(\lg t)^2$ (see Remark 4 (iii) below), while it is not so simple a matter to find a proper asymptotic form of it from Theorem 2. The latter one is given in the next theorem (sharp if $x^2 > t/\lg t$), in which we use the following notation

$$\xi = \frac{|x|}{\sqrt{t}}, \quad \varphi(\alpha) = - \int_1^\infty \frac{e^{-\alpha y}}{y} \lg \left(1 - \frac{1}{y}\right) dy \quad (\alpha > 0),$$

and

$$A_x(t) = \frac{1}{\lg(e^{c_o} t)} \left[1 - \frac{\gamma}{\lg(e^{c_o} t)}\right] \int_{\xi^2/2}^\infty \frac{e^{-u}}{u} du + \frac{\varphi(\xi^2/2)}{[\lg(e^{c_o} t)]^2}.$$

Theorem 3 *Uniformly for $|x| > r_o$, as $t \rightarrow \infty$*

$$P_x[\mathbf{t}_{r_o}^{(2)} \leq t] = A_x(t) + \frac{1}{(\lg t)^3} \times \begin{cases} O(\lg \frac{1}{2} \xi) & \text{for } x^2 < t \\ O((\lg 2\xi)^2/\xi^2) & \text{for } x^2 \geq t. \end{cases} \quad (6)$$

REMARK 4. (i) It holds that $\varphi(\alpha) = O(\alpha^{-1}e^{-\alpha} \log \alpha)$ as $\alpha \rightarrow \infty$ and $\varphi(\alpha) = \frac{1}{6}\pi^2 + \alpha \lg \alpha + O(\alpha)$ as $\alpha \downarrow 0$.

(ii) On using the identity $\int_1^\infty e^{-u} u^{-1} du + \int_0^1 (e^{-u} - 1) u^{-1} du = -\gamma$

$$\int_{\xi^2/2}^\infty \frac{e^{-u}}{u} du = \gamma - \lg(\xi^2/2) - \int_0^{\xi^2/2} \frac{e^{-u} - 1}{u} du.$$

With the help of this together with $2\gamma = \lg[2/e^{c_o} r_o^2]$ it is deduced that as $x^2/t \rightarrow 0$,

$$A_x(t) = 1 - \frac{2 \lg(|x|/r_o)}{\lg(e^{c_o} t)} \left[1 - \frac{\gamma}{\lg(e^{c_o} t)}\right] + \frac{\frac{1}{6}\pi^2 - \gamma^2}{[\lg(e^{c_o} t)]^2} (1 + o(1)) + \frac{\xi^2/2}{\lg(e^{c_o} t)} (1 + O(\xi^2)),$$

which agrees with the expression of $P_x[\mathbf{t}_{r_o}^{(2)} \leq t]$ obtained from Theorem 1 apart from the error of magnitude $O(|\lg \frac{1}{2} \xi|/(\lg t)^3)$ (see the next item).

(iii) Integrating the formula of Theorem 1 leads to

$$P_x[\mathbf{t}_{r_o}^{(2)} > t] = \frac{2 \lg(|x|/r_o)}{\lg(e^{c_o} t)} \left[1 - \frac{\gamma}{\lg(e^{c_o} t)} - \frac{\frac{1}{6}\pi^2 - \gamma^2}{[\lg(e^{c_o} t)]^2} + \dots\right] + O\left(\frac{\xi^2 \lg |x|}{(\lg t)^2}\right) \quad (x^2 < t).$$

The error estimates for $x^2 < t$ in the formula (6) and in this one cannot be improved since otherwise they had become inconsistent as is revealed by examining the sum of their principal terms in comparison to the error terms. On equating the error terms the latter formula is sharper than the former if $\xi^2/\lg \xi = o((\lg t)^{-2})$.

In the dimensions $d \geq 2$ we have the formula

$$E_x[\exp\{-\lambda \mathbf{t}_r^{(d)}\}] = \frac{G_\lambda(|x|, r)}{G_\lambda(r, r)} = \frac{K_{d/2-1}(|x|\sqrt{2\lambda})|x|^{1-d/2}}{K_{d/2-1}(r\sqrt{2\lambda})r^{1-d/2}} \quad (\lambda > 0) \quad (7)$$

(cf. e.g., [3] §7.2, [5]), where G_λ denotes the resolvent kernel for the d -dimensional Bessel process and K_ν the modified Bessel function of order ν . If d is odd, we have a more or less explicit expression of $p_{r, x}(t)$. If d is even, any explicit expression of simple form can not be expected, but an asymptotic estimate of it with the error term whose ratio to $p_{r_o, x}(t)$ is at most $O(\lg t/t)$ is easily obtained. This is also true even if d is not an integer (i.e., for the Bessel processes). Finally we note that Brownian scaling property gives the identity $p_{r/|x|, (1,0)}(t) = x^2 p_{r, x}(x^2 t)$, or, what is the same thing,

$$P_{(1,0)}[\mathbf{t}_{r/|x|}^{(d)} > t] = P_x[\mathbf{t}_r^{(d)} > x^2 t].$$

We prove Theorem 1 in Section 2 and those of Theorems 2 and 3 in Section 3. In the last section some results on $W(\lambda)$ are given.

2 Proof of Theorem 1

Put

$$g(z) = -\lg\left(\frac{1}{2}r_0\sqrt{2z}\right) - \gamma; \quad (8)$$

$$q_x(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{K_0(|x|\sqrt{2z})}{g(z)} e^{tz} dz, \quad (9)$$

and

$$q_x^c(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{K_0(|x|\sqrt{i2u})}{g(iu)} \cos tu \, du. \quad (10)$$

Here $i = \sqrt{-1}$; the integral in (9) is along the imaginary axis; $\lg z$ and \sqrt{z} denote the principal branches in $-\pi < \arg z < \pi$ of the logarithm and the square root, respectively.

Lemma 4 *Uniformly for $|x| > r_0$, as $t \rightarrow \infty$*

$$p_{r_0,x}(t) - q_x(t) = O\left(\frac{1}{t^2(\lg t)} \wedge \frac{1}{|x|^4(1 + \lg^+ |x|)}\right)$$

and the difference $p_{r_0,x}(t) - q_x^c(t)$ admits the same estimate.

Proof. We prove the first relation only, the second one being proved in the same way. By definition

$$K_0(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k}}{(k!)^2} \left(\sum_{m=1}^k \frac{1}{m} - \gamma - \lg(2^{-1}z) \right) \quad (-\pi < \arg z < \pi). \quad (11)$$

It follows that for $-1/2 < u < 1/2$

$$K_0(r_0\sqrt{2iu}) = g(iu) + (r_0/2)^2(2iu)(g(iu) + 1) + O(u^2 \lg |u|); \quad (12)$$

hence

$$\frac{1}{K_0(r_0\sqrt{2iu})} - \frac{1}{g(iu)} = (r_0/2)^2 \frac{-2iu}{g(iu)} \left(1 + \frac{1}{g(iu)} \right) + \frac{O(u^2)}{g(iu)}. \quad (13)$$

If $h(z)$ is holomorphic on $\Re z > 0$, then for any $\xi > 0$

$$\frac{d}{du} h(\xi\sqrt{i2u}) = \frac{1}{2u} k(\xi\sqrt{i2u}) \quad \text{where} \quad k(z) = zh'(z). \quad (14)$$

In view of (13), the identities $K'_0(z) = -K_1(z)$ and $K'_{\nu+1}(z) = -\frac{1}{2}[K_{\nu}(z) + K_{\nu+2}(z)]$ and the asymptotic formula

$$K_{\nu}(z) = (\pi/2z)^{1/2} e^{-z} (1 + O(1/z)) \quad \text{as} \quad |z| \rightarrow \infty \quad (15)$$

($-\pi < \arg z < \pi$, $\nu \geq 0$) (cf. [4] (5.11.9)) (as well as (1)), it therefore suffices to show

$$\int_{-\infty}^{\infty} \frac{2iu}{g(iu)} w(u) K_0(|x|\sqrt{2iu}) e^{itu} du = O\left(\frac{1}{t^2(\lg t)} \wedge \frac{1}{|x|^4(1 + \lg^+ |x|)}\right), \quad (16)$$

where w is a smooth function that equals 1 in a neighborhood of the origin and vanishes outside a finite interval, for the $1 - w$ parts of the integrals in (1) and (9) are $O(e^{-\varepsilon|x|} \wedge t^{-3})$ with $\varepsilon > 0$, of which the bound $O(t^{-3})$ is derived by integrating by parts thrice. In order to

evaluate the integral on the left side of (16) (as well as for the later use) we bring in the two functions

$$G(u) = G_x(u) := K_0(|x|\sqrt{2iu}) \quad \text{and} \quad F(z) = -zK'_0(z) = zK_1(z).$$

Noticing that $G'(u) = -\frac{1}{2u}F(|x|\sqrt{2iu})$ and that $F(z)$, $zF'(z)$ and $z^2F''(z)$ are all bounded (see (30) for more precise estimates), we deduce that

$$|G(u)| \leq C(1 + \lg^+ |u|^{-1}) \quad \text{and} \quad |G^{(j)}(u)| \leq C/|u|^j \quad (j = 1, 2, 3) \quad (17)$$

valid uniformly for $|x| \geq r_\circ$. Integrating by parts transforms the integral in (16) into

$$-\frac{2}{t} \int_{-\infty}^{\infty} \frac{w(u)G(u)}{g(iu)} e^{itu} du - \frac{2}{t} \int_{-\infty}^{\infty} u \frac{d}{du} \left(\frac{w(u)G(u)}{g(iu)} \right) e^{itu} du. \quad (18)$$

In order to evaluate the second integral split its range of integral at $|u| = 1/t$ and to the part $|u| > 1/t$ apply integration by parts once or twice more, which with the help of (17) gives the bound $O(1/t^2 \lg t)$ for the second term of (18) (cf. [7], Lemma 2.2). For the first integral in (18) suppose $x^2 < t$ and consider the function $\tilde{G}(u) := G(u) - g(iu) - \log(r_\circ/|x|)$. Then $\tilde{G}(u)$ is bounded at least for $|u| < 1/\lg t$, $\tilde{G}'(u) = O(1/u)$ and $\tilde{G}''(u) = O(1/u^2)$, and by the same procedure as the one carried out right above we obtain

$$\int_{-\infty}^{\infty} \frac{w(u)\tilde{G}(u)}{g(iu)} e^{itu} du = O\left(\frac{1}{t \lg t}\right).$$

We must estimate the contribution of $g(iu) + \log(r_\circ/|x|)$, but this is easily disposed of since the integral $\int_{-\infty}^{\infty} [-g(iu)]^{-1} e^{itu} du$ equals $AW(e^{c_\circ t})$ for $t > 0$ and $-Ae^{-e^{c_\circ}|t|}$ for $t \leq 0$, where $A = 4\pi e^{c_\circ}$ (see (34) of Appendix), and hence is bounded by $C/|t|(\log t)^2$.

The other bound $O(1/|x|^4 \lg(|x| + 2))$ is obtained by simply scaling the variable u by x^2 and using (15). The proof of the lemma is complete. \square

In view of Lemma 4 we have only to evaluate $q_x(t)$ for the proof of Theorem 1.

Lemma 5 *Uniformly for $|x| > r_\circ$, as $t \rightarrow \infty$*

$$q_x(t) = 2e^{c_\circ} W(e^{c_\circ t}) \lg \frac{|x|}{r_\circ} + O\left(\frac{1 + \lg^+ |x|}{t(\lg t)^2} \left(\frac{x^2}{t} \wedge 1\right)\right).$$

Proof. Put $H(z) = K_0(\sqrt{2z})$, $-\pi < \arg z < \pi$, so that

$$q_x(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{H(x^2 z)}{g(z)} e^{tz} dz.$$

Since $H(z)$ is analytic and bounded by a constant multiple of $-\lg(|z| \wedge \frac{1}{2})$, Cauchy's theorem gives

$$q_x(t) = -\frac{1}{2\pi i} \int_0^\infty \frac{H(-x^2 u + 0i)}{g(-u + 0i)} e^{-tu} du + \frac{1}{2\pi i} \int_0^\infty \frac{H(-x^2 u - 0i)}{g(-u - 0i)} e^{-tu} du. \quad (19)$$

From the formula (11) it follows that for $u > 0$,

$$H(-x^2 u \pm 0i) = \sum_{k=0}^{\infty} (k!)^{-2} \left(-\frac{1}{2} x^2 u\right)^k \left(\sum_{m=1}^k \frac{1}{m} + g\left(-(x/r_\circ)^2 u \pm 0i\right)\right),$$

which combined with the identity $g((x/r_0)^2 z) = -\lg(|x|/r_0) + g(z)$ gives

$$\frac{H(-x^2 u \pm 0i)}{g(-u \pm 0i)} = \frac{1}{g(-u \pm 0i)} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2}x^2 u)^k}{(k!)^2} \left(\sum_{m=1}^k \frac{1}{m} - \lg \frac{|x|}{r_0} \right) + \sum_{k=0}^{\infty} \frac{(-\frac{1}{2}x^2 u)^k}{(k!)^2}.$$

Substitute this expression into the right side of (19). Noticing that the contribution of the last infinite sum cancels out and that since $g(z) = -\frac{1}{2} \lg(e^{-c_0} z)$,

$$\frac{-1}{ig(-u+0i)} + \frac{1}{ig(-u-0i)} = \Im \frac{-2}{g(-u+0i)} = \frac{-4\pi}{(\lg(e^{-c_0} u))^2 + \pi^2},$$

we then deduce that

$$\begin{aligned} q_x(t) &= \frac{1}{2\pi} \int_0^\infty \frac{-4\pi}{(\lg(e^{-c_0} u))^2 + \pi^2} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2}x^2 u)^k}{(k!)^2} \left(\sum_{m=1}^k \frac{1}{m} - \lg \frac{|x|}{r_0} \right) e^{-tu} du \\ &= 2e^{c_0} W(e^{c_0} t) \lg \frac{|x|}{r_0} + \int_0^\infty \frac{-2e^{-tu}}{(\lg(e^{-c_0} u))^2 + \pi^2} \sum_{k=1}^{\infty} \frac{(-\frac{1}{2}x^2 u)^k}{(k!)^2} \left(\sum_{m=1}^k \frac{1}{m} - \lg \frac{|x|}{r_0} \right) du \\ &= 2e^{c_0} W(e^{c_0} t) \lg \frac{|x|}{r_0} + O\left(\frac{(\lg(|x|+2))}{t(\lg t)^2} \left(\frac{x^2}{t} \wedge 1\right)\right) \quad (t \rightarrow \infty) \end{aligned}$$

uniformly for $|x| > r_0$ as desired. Here for the last equality we have applied the crude bounds

$$\sum_{k=1}^{\infty} \frac{(-y)^k}{(k!)^2} = O(y \wedge 1) \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{(-y)^k}{(k!)^2} \sum_{m=1}^k \frac{1}{m} = O(y \wedge 1) \quad (y > 0),$$

which are readily verified by using $K_0(\sqrt{-4y}) = O(e^{-2\sqrt{y}})$ ($y > 1$) together with the formulae (11) and

$$\sum_{k=0}^{\infty} \frac{(-y)^k}{(k!)^2} = J_0(2\sqrt{y}) = \frac{2}{\pi} \int_0^{\pi/2} \cos(2\sqrt{y} \sin \theta) d\theta = O(y^{-1/4}) \quad (y > 1)$$

(J_0 is the Bessel function of first kind of order 0). Thus (3) has been verified. \square

3 Proof of Theorems 2 and 3

For the present purpose we estimate the function q_x^c that is defined by (10). Theorem 3 will be proved by computing

$$Q_x(t) := \int_0^t q_x^c(s) ds = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{K_0(|x|\sqrt{i2u})}{g(iu)} \cdot \frac{\sin tu}{u} du.$$

Owing to the fact noted in Remark 4 (iii) we have only to consider the case $x^2 > t/(\log t)^2$, when the difference between $Q_x(t)$ and $\int_0^t p_{r_0,x}(u) du$ is negligible in view of Lemma 4.

The computation is based on the following formula ([4] (5.10.25)):

$$K_0(|x|\sqrt{i2u}) = \frac{1}{2} \int_0^\infty \exp\left(-y - i\frac{x^2}{2y}u\right) \frac{dy}{y}. \quad (20)$$

Let $\xi = |x|/\sqrt{t}$ and $\varphi(\alpha) = -\int_1^\infty e^{-\alpha y} y^{-1} \lg[1-y^{-1}] dy$, $\alpha > 0$ as in Introduction and define

$$R(t) = \frac{-2}{\pi[\lg(e^{c_0} t)]^2} \int_{-\infty}^{\infty} K_0(\xi\sqrt{i2u}) \frac{[\lg(iu)]^2}{\lg(iu/e^{c_0} t)} \cdot \frac{\sin u}{u} du.$$

Note that $\varphi(0+) = -\int_0^1 u^{-1} \lg(1-u) du = \pi^2/6$, in particular φ is bounded.

Lemma 6 $Q_x(t) = A(t) + R(t)$, where

$$A(t) := \frac{1}{\lg(e^{c_0}t)} \left[1 - \frac{\gamma}{\lg(e^{c_0}t)} \right] \int_{\xi^2/2}^{\infty} \frac{e^{-y}}{y} dy + \frac{\varphi(\xi^2/2)}{[\lg(e^{c_0}t)]^2}. \quad (21)$$

Proof. We let $c_0 = 0$ for simplicity and write $Q_x(t)$ in the form

$$Q_x(t) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{K_0(\xi\sqrt{i2u})}{g(iu/t)} \frac{\sin u}{u} du.$$

Since $g(z) = -\frac{1}{2} \lg(z/e^{c_0})$, upon noting $\frac{1}{1-x} = 1 + x + \frac{x^2}{1-x}$

$$\begin{aligned} \frac{1}{2g(iu/t)} &= \frac{1}{-\lg(iu/t)} = \frac{1}{\lg t} \cdot \frac{1}{1 - \lg(iu)/\lg t} \\ &= \frac{1}{\lg t} \left[1 + \frac{\lg(iu)}{\lg t} \right] - \frac{1}{\lg(iu/t)} \left[\frac{\lg(iu)}{\lg t} \right]^2. \end{aligned}$$

Hence

$$Q_x(t) = \frac{I(t)}{\lg t} + \frac{II(t)}{(\lg t)^2} + R(t), \quad (22)$$

where

$$I(t) = \frac{2}{\pi} \int_{-\infty}^{\infty} K_0(\xi\sqrt{i2u}) \frac{\sin u}{u} du \quad \text{and} \quad II(t) = \frac{2}{\pi} \int_{-\infty}^{\infty} \lg(iu) K_0(\xi\sqrt{i2u}) \frac{\sin u}{u} du.$$

Substitution from (20) gives

$$I(t) = \frac{2}{\pi} \int_0^{\infty} \frac{\sin u}{u} du \int_0^{\infty} \frac{e^{-y}}{y} \cos \frac{\xi^2 u}{2y} dy.$$

Since

$$\int_0^{\infty} \frac{\sin u}{u} \cos \frac{\xi^2 u}{2y} du = \begin{cases} \pi/2 & y > \frac{1}{2}\xi^2, \\ 0 & y < \frac{1}{2}\xi^2, \end{cases}$$

by interchanging the order of integration (easily justified) we obtain

$$I(t) = \int_{\xi^2/2}^{\infty} \frac{e^{-y}}{y} dy.$$

Similarly

$$II(t) = \frac{1}{\pi} \int_0^{\infty} \frac{e^{-y}}{y} dy \int_{-\infty}^{\infty} \frac{\sin u}{u} \exp\left(\frac{-i\xi^2}{2y}u\right) \lg(iu) du.$$

We write the inner integral on the right side in the form

$$\int_{-i\infty}^{i\infty} \frac{e^{(1-s)z} - e^{-(1+s)z}}{i2z} \lg z dz, \quad s = \frac{\xi^2}{2y}$$

and prove that this integral equals

$$\begin{cases} 0 & \text{if } s > 1, \\ -\pi(\gamma + \lg(1-s)) & \text{if } s < 1. \end{cases} \quad (23)$$

To this end first consider the case $s < 1$. We apply Cauchy's theorem by considering the term involving $e^{(1-s)z}$ in the left half plane and the other in the right half. Note that the

integrand is discontinuous along the negative real line but regular otherwise. Let $C^+(\varepsilon)$ and $C^-(\varepsilon)$ denote the right and left halves of the circle $z = \varepsilon e^{i\theta}$ for small $\varepsilon > 0$. We then observe that the integral in question may be written as

$$\left[- \int_{-\infty+0i}^{-\varepsilon+0i} + \int_{-\infty-0i}^{-\varepsilon-0i} + \int_{C^-(\varepsilon)} \right] \frac{e^{(1-s)z}}{i2z} \lg z \, dz + \int_{C^+(\varepsilon)} \frac{e^{-(1+s)z}}{i2z} \lg z \, dz$$

Noting that $\lg(-u \pm 0i) = \lg u \pm i\pi$ for $u > 0$ we rewrite this in the form

$$\pi \int_{\varepsilon}^{\infty} \frac{e^{-(1-s)u}}{u} \log u \, du + \frac{1}{2} \int_{-\pi}^{\pi} (\lg \varepsilon + i\theta) d\theta + O(\varepsilon \lg \varepsilon).$$

On letting $\varepsilon \downarrow 0$ with the help of the identity $\lim_{\varepsilon \downarrow 0} [\int_{\varepsilon}^{\infty} e^{-au} u^{-1} du + \lg \varepsilon] = -\gamma - \lg a$ ($a > 0$) this is reduced to the second expression of (23). The case $s > 1$ would now be obvious.

Now one can conclude that

$$\begin{aligned} II(t) &= -\gamma \int_{\xi^2/2}^{\infty} \frac{e^{-y}}{y} dy - \int_{\xi^2/2}^{\infty} \frac{e^{-y}}{y} \lg \frac{y - \frac{1}{2}\xi^2}{y} dy \\ &= -\gamma \int_{\xi^2/2}^{\infty} \frac{e^{-y}}{y} dy + \varphi(\xi^2/2), \end{aligned} \quad (24)$$

and substitution into (22) completes the proof of the lemma. \square

Proof of Theorem 3. We must evaluate $R(t)$. We claim that

$$R(t) = \begin{cases} O(|\lg \frac{1}{2}\xi|/(\lg t)^3) & \text{if } \xi < 1, \\ O((\lg(2\xi))^2/\xi^2(\lg t)^3) & \text{if } \xi \geq 1, \end{cases} \quad (25)$$

which is enough for the proof of Theorem 3.

First consider the case $\xi < 1$. We split the integral defining $R(t)$ at $|u| = 1$ and $|u| = 1/\xi^2$. For simplicity we consider only the integral on $u > 0$ and let $c_0 = 0$. Let $D(1) = (0, 1)$, $D(2) = (1, 1/\xi^2]$, $D(3) = (1/\xi^2, \infty)$ and put

$$B_j = \int_{D(j)} K_0(\xi\sqrt{i2u}) \frac{[\lg(iu)]^2}{\lg(iu/t)} \cdot \frac{\sin u}{u} du \quad (j = 1, 2, 3). \quad (26)$$

In view of the bound $K_0(z) = O(\lg \frac{1}{4}|z|)$ ($|z| < 1$) we immediately obtain that $|B_1| = O(|\lg \frac{1}{2}\xi|/\lg t)$. For B_2 perform integration by parts (with $\sin u$ to be integrated). Using the identity (14) with $|x|$ replaced by ξ we then infer that

$$\begin{aligned} |B_2| \leq \frac{C}{\lg t} + C \int_1^{1/\xi^2} & \left[\frac{|F(\xi\sqrt{i2u})| \cdot |\lg(iu)|^2}{|\lg(iu/t)|} \right. \\ & \left. + |K_0(\xi\sqrt{i2u})| \frac{(|\lg(iu/t)| + 1) \cdot |\lg(iu)|^2}{|\lg(iu/t)|^2} \right] \frac{du}{u^2}. \end{aligned} \quad (27)$$

Since $F(z) = -zK_1(z) = O(1)$ ($|z| < 1$), this shows that $|B_2| = O(|\lg \frac{1}{2}\xi|/\lg t)$. On changing the variable of integration

$$B_3 = \int_1^{\infty} K_0(\sqrt{i2u}) \frac{[\lg(iu/\xi^2)]^2}{\lg(iu/|x|)} \cdot \frac{\sin u/\xi^2}{u} du. \quad (28)$$

The integration by parts then gives $|B_3| \leq C\xi^2 |\lg \xi|^2 / |\lg |x|| = O(|\lg \frac{1}{2}\xi|/\lg t)$. Thus the claim is verified in the case $\xi < 1$.

For the case $\xi \geq 1$ we make a similar argument but without integration by parts. Employing the bound given in (15) for $|u| \geq 1/\xi^2$, we obtain

$$|R(t)| \leq \frac{C}{|\lg t|^2} \left[\int_0^{1/\xi^2} \frac{|\lg(\frac{1}{2}\xi\sqrt{u})| \cdot |\lg(iu)|^2}{|\lg(iu/t)|} du + \int_{1/\xi^2}^\infty \frac{e^{-\xi\sqrt{u}} |\lg(iu)|^2}{(\xi\sqrt{u})^{1/2} |\lg(iu/t)|} \cdot \frac{|\sin u|}{|u|} du \right]$$

The first integral in the square brackets and the part \int_{1/ξ^2}^∞ of the second one are easily evaluated to be $O((\lg \xi)^2/\xi^2 \lg t)$. For the part \int_0^{1/ξ^2} we first replace $|\lg(iu/t)|$ in the denominator by $\lg t$ and then make change of the variable by $y = \xi^2 u$ to find it to be $O((\lg \xi)^2/\xi^2 \lg t)$. Hence $R(t) = O((\lg \xi)^2/\xi^2 (\lg t)^3)$, as required. \square

We are to derive Theorem 2 from Theorem 3 by evaluating the derivatives $A'(t)$ and $R'(t)$.

Lemma 7 *There exists a constant C such that for $t > 2$ $|R'(t)| \leq C/t(\lg t)^3$ for $x^2 < t$ and $|R'(t)| \leq C[1 + (\lg \xi)^2]/|x|^2(\lg t)^3$ for $x^2 \geq t$.*

Proof. As before let $c_0 = 0$ for simplicity. Differentiate the defining expression of $R(t)$ and observe that in the integrand there then appears

$$K'_0(\xi\sqrt{i2u})\sqrt{i2u}\xi' = -K'_0(\xi\sqrt{i2u})\sqrt{i2u}\xi/2t.$$

With the function $F(z) = -zK'_0(z) = zK_1(z)$, we can write the result as

$$\begin{aligned} R'(t) &= \frac{-1}{\pi[\lg t]^2 t} \int_{-\infty}^\infty F(\xi\sqrt{i2u}) \frac{[\lg(iu)]^2}{\lg(iu/t)} \frac{\sin u}{u} du \\ &\quad + \frac{-2}{\pi[\lg t]^2 t} \int_{-\infty}^\infty K_0(\xi\sqrt{i2u}) \frac{[\lg(iu)]^2}{[\lg(iu/t)]^2} \frac{\sin u}{u} du \\ &\quad + \frac{4}{\pi[\lg t]^3 t} \int_{-\infty}^\infty K_0(\xi\sqrt{i2u}) \frac{[\lg(iu)]^2}{\lg(iu/t)} \frac{\sin u}{u} du. \end{aligned} \quad (29)$$

Consider the case $\xi < 1$. By the same argument that derives the bound of B_j in the proof of Theorem 3 the last two terms are both $O((\lg \xi)/t(\lg t)^4)$. It holds that

$$F(z) = 1 + O(z^2 \lg |z|) \quad \text{and} \quad F'(z) = z \lg z + O(z) \quad (|z| < 1) \quad (30)$$

(as being readily deduced from (11)); and $F'(z) = K_1(z) - 2^{-1}z[K_0(z) + K_2(z)] = O(|z|^{1/2}e^{-z})$ ($|z| > 1/2$). Obviously the first integral in (29) restricted on $|u| < 1$ is $O(1/\log t)$. The integral on $1 < u < 1/\xi^2$ is also $O(1/\log t)$ as is deduced in the same way as B_2 is estimated (one has only to replace in (27) K_0 and F by F and zF' , respectively). Similarly the integral on $1/\xi^2 < u < \infty$ is evaluated to be at most $O(1/\log t)$ (one may replace K_0 by F in (28)). Hence $R'(t) = O(1/(\lg t)^3 t)$ as $t \rightarrow \infty$ uniformly for $\xi < 1$.

In the case $\xi \geq 1$ one uses the bound $F(\xi\sqrt{i2u}) = O(e^{-\xi\sqrt{|u|/2}})$ to see that the integral involving F is at most a constant multiple of

$$\frac{1}{\lg t} \int_0^1 e^{-\frac{1}{2}\xi s} (\lg s)^2 s ds = O\left(\frac{1 + (\lg \xi)^2}{\xi^2 \lg t}\right),$$

the integral on $|u| > 1$ being much smaller than this for large ξ . The other integrals are estimated in a similar way. \square

Proof of Theorem 2. Recalling $\gamma = -\int_0^\infty e^{-y} \lg y dy$, observe that

$$\varphi'(\alpha) = -(\gamma + \lg \alpha) \frac{e^{-\alpha}}{\alpha} - \frac{1}{\alpha} \int_\alpha^\infty \frac{e^{-y}}{y} dy.$$

Then some easy computation leads to

$$A'(t) = \frac{\lg(\frac{1}{2}e^{c_0}x^2)}{t[\lg(e^{c_0}t)]^2} e^{-x^2/2t} + \frac{2\gamma}{t[\lg(e^{c_0}t)]^3} \int_{\xi^2/2}^\infty \frac{e^{-u}}{u} du - \frac{2\varphi(\xi^2/2)}{t[\lg(e^{c_0}t)]^3}. \quad (31)$$

where $A(t)$ is defined by (21). Considering the case $x^2 < t$ let $b(t)$ denote the sum of the first two terms on the right side of (5), namely

$$b(t) = \frac{\lg(\frac{1}{2}e^{c_0}x^2)}{t[\lg(e^{c_0}t)]^2} e^{-x^2/2t} + \frac{2\gamma \lg^+(t/x^2)}{t(\lg t)^3},$$

to obtain

$$A'(t) = b(t) + O\left(\frac{1}{t(\lg t)^3}\right). \quad (32)$$

Combined with Lemma 7 this shows Theorem 2 in the case $\xi < 1$. The case $\xi \geq 1$ is also deduced from (31) in a similar way. \square

REMARK 5. (i) Let $c_0 = 0$. Then computations similar to those performed for derivation of (24) lead to

$$q_x(t) = \int_1^\infty \frac{e^{-\frac{1}{2}\xi^2/y}}{y} e^{-(y-1)t} dy - \int_0^1 \frac{e^{-\xi^2/2y}}{y} dy \int_0^\infty \frac{e^{-(1-y)tu}}{(\lg u)^2 + \pi^2} du \quad (33)$$

and

$$Q_x(t) = \int_0^1 \frac{e^{-\xi^2/2y}}{y} dy \int_0^\infty \frac{e^{-(1-y)u}}{([\lg(u/t)]^2 + \pi^2)u} du + S_x(t)$$

with $S_x(t) = -\int_1^\infty \frac{e^{-\xi^2/2y}}{y} e^{-(y-1)t} dy + e^{-t} \int_0^\infty \frac{e^{-y}}{y} e^{-x^2/2y} dy = O(1/t)$ uniformly in $|x| > r_0$. (In the derivation of (33) we need to interchange the order of the repeated integral that arises as one substitutes (20) into the defining expression of $q_x(t)$. The argument for justification of the interchange involves somewhat delicate analysis.) Although it seems hard to derive Theorems 2 or 3 from these expressions, they are useful if ξ is large: at least we can derive from (33) that uniformly for $|x| \lg |x| < t < x^2$, as $|x| \rightarrow \infty$

$$q_x(t) = \frac{e^{-x^2/2t}}{t \lg t} (1 + o(1)).$$

(ii) If one uses the representation

$$K_0(|x|\sqrt{2iu}) = \frac{1}{\pi} \int_0^\pi d\alpha \int_0^\infty \frac{\cos(|x|r \sin \alpha)}{2iu + r^2} r dr$$

([2], p.45 (15))) in place of (20), certain computations using Cauchy's theorem lead to a result similar to Theorem 2, but this way is more involved than that we have adopted.

4 Appendix

(A) In order to derive the expansion (4) observe

$$\lambda W(\lambda) = \frac{-1}{\pi} \Im \int_0^\infty \frac{e^{-\lambda u} \lambda du}{\lg u + i\pi} = \frac{1}{\pi \lg \lambda} \int_0^\infty \Im \left(1 - \frac{\lg u + i\pi}{\lg \lambda} \right)^{-1} e^{-u} du.$$

On using the formula $\int_0^\infty (\lg u)^2 e^{-u} du = \frac{1}{6}\pi^2 + \gamma^2$ ([2] p.169 (13)) a standard argument then leads to (4). By a more sophisticated method Bouwkamp [1] derives an asymptotic expansion for a class of functions defined by similar Laplace transforms; as a special case it gives that $\lambda W(\lambda) = \sum_{n=0}^\infty c_n (\log \lambda)^{-n-1}$, with (c_n) determined by

$$\sum_{n=0}^\infty \frac{c_n}{n!} z^n = \frac{z}{\Gamma(1-z)}.$$

(Although in [1] only the case $\lambda \rightarrow \infty$ is considered, the method is valid for the case $\lambda \downarrow 0$.)

(B) Here we derive three Fourier representations of $W(\lambda)$. The first one is

$$\frac{1}{2\pi} \int_{-\infty}^\infty \frac{e^{-i\lambda u}}{\lg(-iu)} du = \begin{cases} W(\lambda) & (\lambda > 0), \\ -e^\lambda & (\lambda < 0). \end{cases} \quad (34)$$

For $\lambda > 0$ this is obtained by Cauchy's theorem with the help of the equality $\lg(-(u \pm 0i)) = \lg u \mp i\pi$ valid for $u > 0$. On the other hand, by calculus of residues we see that for $\lambda < 0$,

$$\frac{1}{2\pi} \int_{-\infty}^\infty \frac{e^{-i\lambda t}}{\lg(-it)} dt = \frac{-1}{2\pi} \int_{|z|=1} \frac{e^{\lambda z}}{\lg z} \frac{dz}{i} = -e^\lambda.$$

It is noted that since $\lg(-it) = \lg|t| - i\frac{1}{2}\pi \operatorname{sgn} t$, the last identity means

$$\frac{1}{2} \int_0^\infty \frac{\sin |\lambda| t}{[\lg t]^2 + \frac{1}{4}\pi^2} dt = \frac{1}{\pi} \int_0^\infty \frac{\lg t}{[\lg t]^2 + \frac{1}{4}\pi^2} \cos \lambda t dt + e^{-|\lambda|},$$

which together with (34) leads to

$$W(\lambda) = \int_0^\infty \frac{\sin \lambda t}{[\lg t]^2 + \frac{1}{4}\pi^2} dt - e^{-\lambda} = \frac{2}{\pi} \int_0^\infty \frac{(\lg t) \cos \lambda t}{[\lg t]^2 + \frac{1}{4}\pi^2} dt + e^{-\lambda} \quad (\lambda > 0). \quad (35)$$

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